

Folding Corners of the Habits of Mind

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Students fold paper to make and test conjectures while reasoning about and discussing geometric ideas.

Key Words: Geometry, Habits of mind, Visual thinking, Preservice teachers

Reasoning is central to what Cuoco, Goldenberg, and Mark refer to as mathematical habits of mind (1996). “Reasoning is an integral part of doing mathematics. Students should enter the middle grades with the view that mathematics involves examining patterns and noting regularities, making conjectures about possible generalizations, and evaluating the conjectures” (NCTM 2000, p. 262). Mathematical habits of mind are the practices that mathematicians use in their work, such as searching for patterns, experimenting, tinkering, describing, inventing, visualizing, and conjecturing. Teachers who wish to promote these habits in their students, however, are faced with the daunting challenge of finding tasks that can develop and further such habits.

A paper-folding task can be an effective vehicle for such mathematical reasoning. When one thinks of origami and its connections to mathematics, images of intricate folded geometric models come to mind. In this capacity, origami has great potential for enriching mathematics classrooms (Cipoletti and Wilson 2004; DeYoung 2009; Higginson and Colgan 2001). When building an origami model, however, the person who is folding the paper is typically following a set procedure. The important mathematical thinking resides in the reasoning about the model, not the folding process. What follows demonstrates a novel pedagogical approach to origami and how habits of mind can serve as a foundation for mathematical exploration.

GEOMETRIC HABITS OF MIND

Habits of mind, as originally defined, should permeate all mathematics instruction. The framework, originally applied to high school curricula, is equally applicable to the middle grades and elementary school (Goldenberg, Mark, and Cuoco 2010; Mark et al. 2010). One can easily see these ideas being made explicit in the Standards for Mathematical Practice (CCSS 2010), which represent the latest move toward specifying how students should experience mathematical ideas. Although general habits of mind, or mathematical practices, are important, it is also helpful to focus on those habits of mind that are specialized to the mathematical concepts being studied. Driscoll (2007) identified four geometric habits of mind (GHoM) that are central to the study of geometry:

1. Reasoning with relationships
2. Generalizing geometric ideas

3. Investigating invariants
4. Balancing exploration and reflection

The first GHoM, reasoning with relationships, concerns identifying key properties in objects and then exploring relationships between properties and the geometric elements in the problem. The second GHoM, generalizing geometric ideas, can be seen when students explore a geometric problem and use insights to better understand related or larger problems. Students with this habit of mind ask themselves how conjectures about particular situations might generalize to larger classes of geometric objects. The third GHoM, investigating invariants, involves actively transforming a geometric object and reflecting on what aspects of the situation remain the same in the midst of that change. The last GHoM, balancing exploration and reflection, involves turning one's thinking inward while exploring a model or situation. Students not only continually take stock of their insights but also actively think about how those insights relate to the problem statement.

TURNED-UP-POINT ACTIVITY

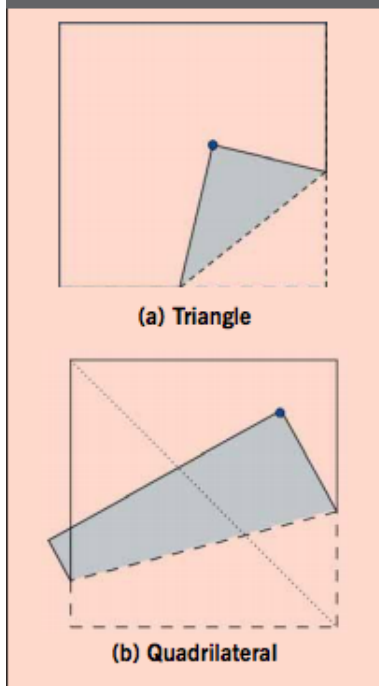
A paper-folding activity, created by Kazuo Haga (2008), was posed to a class of preservice teachers in a geometry course for elementary school and middle school teachers. Haga called the paper-folding process origamics, in contrast to traditional origami. This process moved beyond simply creating paper models and producing a final product to a process of folding and unfolding paper to highlight invariant properties. Origamics-related problems often involved simple folds that revealed surprisingly rich geometric relationships.

The version of Haga's problem used with students was referred to as a turned-up-point (TUP) problem by Hull (2006). Hull described both the mathematical ideas imbedded in the exploration as well as notes on their pedagogical uses. Hull's *Project Origami: Activities for Exploring Mathematics* and Haga's *Origamics: Mathematical Explorations through Paper Folding* detailed several other explorations in origamics that could be transformed into rich classroom activities. The turned-up-point, or TUP, problem involves folding the corner of a piece of paper and examining the shape of the flap that is formed. More precisely, begin with a square piece of paper and pick a point anywhere in the interior of the paper. Fold the paper so that the bottom right vertex falls on this point (see fig. 1). Focus attention on the flap that is folded up (the shaded portions in fig. 1) and determine how many sides that polygonal flap has. Repeat this process several times with different interior points to answer the following two questions:

1. What is the possible number of sides of the polygonal flap?
2. How can you predict the number of sides that the flap will have, viewing only the location of the point on the paper?

I encourage the reader to get a square piece of paper and try this before moving on.

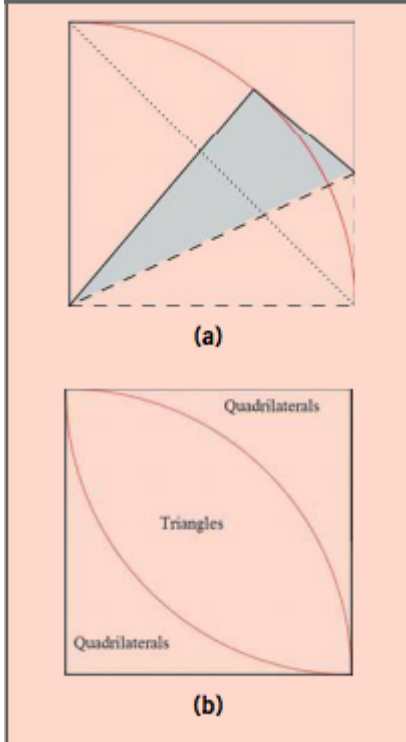
Fig. 1 Folding a corner of a square piece of paper to different points results in shapes that are worth noting.



After some exploration, it becomes apparent that the flap can be a triangle or a quadrilateral only. The only way to form a pentagonal flap is to extend the bottom-right corner off the paper. The focus of the exploration, then, is to determine precisely when the flap is a triangle and when it is a quadrilateral. Figure 1b shows a fold that results in a quadrilateral. Notice that this quadrilateral flap is formed precisely because the creased fold intersects the left edge of the paper rather than the bottom edge. Similarly, based on the symmetry of the square, choosing a point to the left of the diagonal in figure 1b will result in a quadrilateral precisely when the fold intersects the top edge of the paper.

The key to finding where the transition from a triangle to a quadrilateral will occur is to note all those points where the folded crease intersects the bottom-left vertex. These points form a circular arc centered at the lower-left vertex whose radius is equal to the length of the side of the square (see figure 2a). Repeat this process for points on the left of the diagonal to form the boundary that separates a triangle from a quadrilateral (see figure 2b).

Fig. 2 Folding a corner to various points yields a triangle and folding to other points yields a quadrilateral. The boundaries between these two sets of points are sectors of a circle whose center is the corner of the paper and whose radius is the length of the sides.



OPPORTUNITIES FOR DEVELOPING HABITS OF MIND

This paper-folding task had several mathematically beneficial features:

1. It required no prerequisite paperfolding experience. The preservice teachers did not have to learn techniques or approaches associated with more complicated origami constructions.
2. It had a low barrier to entry for mathematical knowledge. Preservice teachers at different levels could immediately begin work.
3. It was exploratory by design and offered numerous opportunities for discussion. Beyond these general features, the task also offered opportunities for the preservice teachers to engage in each of the four geometric habits of mind described earlier. The following sections highlight aspects of their work and showcase the ways that the TUP problem was effective in developing habits of mind.

INVESTIGATING IN VARIANTS

The TUP problem statement was essentially a question about invariance. To predict which points resulted in triangle flaps, for instance, the preservice teachers could not actually test them all. Instead, they needed to identify those characteristics that these points shared. They identified what stayed the same about the paper for the flap to remain a triangle and what changed to shift to a quadrilateral.

All the preservice teachers began the exploration with a form of empirical investigation. Some began folding the paper repeatedly; others took a more systematic approach and marked points that they tested (see fig. 3). After the teachers had tested several points, they were asked to articulate what they thought was true about the triangle points. This required them to shift their focus from empirical testing to making conjectures.

Fig. 3 Some participants tested various destination points for the folded-up corner to predict the pattern when a triangle and when a quadrilateral were formed by the flap.



The initial conjectures were often imprecise. For example, some preservice teachers noted that triangles were “on the inside” and quadrilaterals were “close to the edge.” Although imprecise, these statements marked an important change in students’ reasoning. I asked one preservice teacher, Sarah, about an oval figure she had drawn on the paper. She explained that the figure was the “boundary between triangle land and quadrilateral land.” Her explanations indicated to me that she was no longer testing individual points. She was focusing her attention on the boundary and how the flap was transitioning dynamically from one shape to another. This process of noticing regularity in the midst of change is the essence of investigating invariance.

BALANCING EXPLORATION AND REFLECTION

As mentioned earlier, this problem had a very low barrier to entry. The preservice teachers could begin folding and testing points right away. To solve this problem,

however, they had to recognize when to stop folding and when to begin reflecting on the information that was collected. This is the central nature of Driscoll's third GHoM: balancing exploration and reflection.

Although the preservice teachers began working easily, it was difficult for many of them to recognize when there was enough information to start analyzing. Rather than simply telling them when to stop folding, I instead asked what they were noticing. Those who could not verbalize their thinking were encouraged to pay close attention to what they were noticing while making a few more folds.

The most effective way to shift the preservice teachers' focus from exploration to reflection, however, was through whole-class discussions. Because it was difficult for students to directly refer to the marks on their small paper in a way that everyone could see, they were forced to put their ideas into words. It also shifted the discussion to the mathematical relationships they were noticing rather than the actual folds they had made. One strength of the TUP problem was that it forced the class to regulate their exploration and find a balance between it and careful reflection.

REASONING WITH RELATIONSHIPS

For the preservice teachers to explain their ideas in mathematically precise ways, their attention had to change. Rather than accumulating points that generated triangles or quadrilaterals, they were now required to reason about the relationships they were identifying. This was the essence of the second GHoM: generalizing geometric ideas.

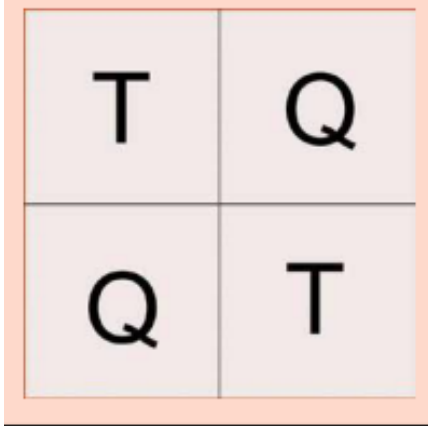
Several preservice teachers tried to reason about their conjectures by trying to identify relationships between the points and the symmetries of the square. When using square paper, it was natural to fold the paper in half either vertically or diagonally, highlighting lines of symmetry. Some conjectured that the quadrants formed by the perpendicular bisectors separated the triangle points from the quadrilateral points (see fig. 4). Although this conjecture was not true, it did indicate that they were actively searching for relationships within the square.

Exploring the diagonal lines of symmetry for the square was more productive. Jessica, a preservice teacher, was excited to share with the class her discovery that the flap was always a triangle along one of the diagonals but that it shifted between triangles and quadrilaterals along the other diagonal. Others in the class chimed in, realizing that the first diagonal highlighted symmetry in the problem and that they needed to focus on only one side of it.

For many other preservice teachers, it was apparent that the boundary between points that yielded triangles and those that yielded quadrilaterals was some kind of curve. Much like Sarah's designation of "triangle land," several people also sketched possible curves based on their empirical evidence but could not describe the curve mathematically. An important realization occurred when Jessica reflected on her

discovery about the diagonal. She demonstrated how, when you slowly moved the flap, the transition occurred when the side of the flap was the entire bottom edge of the paper. After further class discussion, Eric demonstrated rotating that flap along the circular arc to show the whole boundary. Jessica's suggestion to focus on the relationship between the flap and the edge of the square was central to generating a consensus about the solution.

Fig. 4 Some students incorrectly identified the regions for the folded-up point and the shape of the flap that resulted. T represented a triangle, and Q represented a quadrilateral.



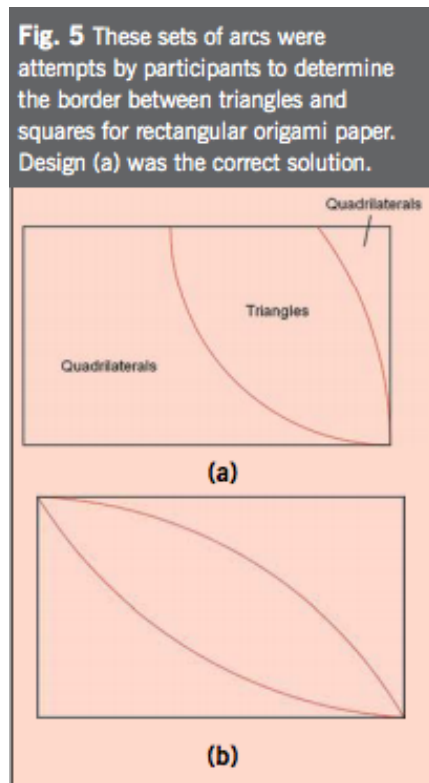
GENERALIZING GEOMETRIC IDEAS

Discovering a solution often marks the end of a mathematical exploration. According to the GHoM framework, however, we also want to instill an inclination to wonder about how the solution generalizes to new situations. After the preservice teachers found the solution, the class discussed the essential features of the problem itself. What were the unique features of this problem? It involved a square piece of paper, folding a corner to an interior point on the paper, and looking at the shape of the flap. I then asked the class to consider these features and decide if one or more of them could be changed to create a new problem. One appropriate response was that one might start with a general rectangular piece of paper or investigate folding to points that are not on the paper.

The process of generalization was quite difficult for the preservice teachers. Identifying the central elements of the problem and brainstorming possible modifications took much effort, almost as much as solving the initial problem. By doing this, however, they had to decide what constituted a natural generalization of the original problem. They also had to use their intuition to decide which new problem would be interesting. Last, they needed to reflect on how elements of the solution to the original problem generalized to the new situation, for instance, thinking about how circles might play a role.

As a class, they decided to explore an extension in which the paper was a general rectangle rather than a square. The solution to this new problem relies on the same relationships discovered in the original problem. As before, circular arcs will form the boundary between the regions. This time, however, the circles have different radii (see fig. 5a).

With the new paper shape, the preservice teachers repeatedly folded the paper, gathering empirical data as with the square paper. When they were encouraged to reflect on the solution using square paper, there were definite signs of generalization. Some students folded and formed a region on the rectangular paper congruent to the one found with the square paper in which the circle's radius was the length of the short side. Others drew curved regions between the opposite vertices, reminiscent of the solution with the square (see fig. 5b). Only after careful exploration and further classroom discussion did the correct relationships finally become clear.



THE BROAD PICTURE

Activities such as the TUP paperfolding exploration are enriching tasks because they can motivate and interest students. However, they can also serve an even deeper purpose. By focusing on geometric habits of mind, our students not only have opportunities to explore important geometric ideas but also learn how to test ideas, make conjectures, pose new questions, and feel the thrill of uncovering

relationships that appear, seemingly, out of nowhere. These ideas are, by no means, limited to paper folding. I encourage you, the reader, to consider how these ideas might be similarly infused in your geometry lessons as well.

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