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The two variable substitution problem for free products of groups

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Abstract. We consider equations of the form $W(x, y) = U$ with U an element of a free product G of groups. We show that with suitable algorithmic conditions on the free factors of G, one can effectively determine whether or not the equations have solutions in G. We also show that under certain hypotheses on the free factors of G and the equation itself, the equation $W(x, y) = U$ has only finitely many solutions, up to the action of the stabilizer of $W(x, y)$ in $\mathrm{Aut}(\langle x, y; \rangle)$.

1 Introduction

If G is a group with specified finite generating set $\mathscr{A}$ and $W(x_1, \ldots, x_n)$ is an element of the free group $F = \langle x_1, x_2, \ldots, x_n; \rangle$, the *substitution problem* for W in G is the problem of deciding, for words U on $\mathscr{A} \cup \mathscr{A}^{-1}$, if there is an n-tuple $(X_1, X_2, \ldots, X_n)$ of words on $\mathscr{A} \cup \mathscr{A}^{-1}$ such that $W(X_1, X_2, \ldots, X_n) = U$ in G, that is, of determining if there is a homomorphism $\phi : F \to G$ with $W\phi = U$ in G. In this paper, we shall consider particularly the *two variable substitution problem*, that of deciding, for $W(x, y) \in F = \langle x, y; \rangle$ and U a word on $\mathscr{A} \cup \mathscr{A}^{-1}$, if

$$W(x, y) = U \tag{1}$$

has a solution. Our result, Theorem 1 below, is that the two variable substitution problem is solvable for a free product $G = *_{i \in I} G_i$ of finitely generated, torsion-free groups provided that this problem and certain other algorithmic problems are solvable in the free factors. In the course of proving Theorem 1, we establish another result, Theorem 2, which says that under certain hypotheses, equation (1) has only finitely many orbits of solutions under the action of the stabilizer of $W(x, y)$ in $\mathrm{Aut}(\langle x, y; \rangle)$.

There is a hypothesis to be used in Theorem 1 that requires definition. We say that a group G with a specified finite generating set $\mathscr{A}$ has *effective extraction of roots* if, for each word U on $\mathscr{A} \cup \mathscr{A}^{-1}$ with $U \neq 1$ in G, we can effectively calculate a finite set of pairs $\{(V_1, n_1), \ldots, (V_k, n_k)\}$ with each V_i a word on $\mathscr{A} \cup \mathscr{A}^{-1}$ and $n_i \in \mathbb{Z}$ such that every solution to $x^n = U$ with $x \in G$ and $n \in \mathbb{Z}$ is of the form $x = V_j$, $n = n_j$ for some j, $1 \leqslant j \leqslant k$. Among groups satisfying this condition are free groups and torsion-free hyperbolic groups (cf. [4] and [3, Corollary 7.2]). Note that any group with effective extraction of roots must be torsion-free.

Theorem 1. *Let $G = *_{i \in I} G_i$ be a free product of finitely generated groups G_i, where each G_i has solvable word problem, solvable conjugacy problem, effective extraction of roots, and solvable two variable substitution problem. Then the two variable substitution problem for G is solvable.*

This is a generalization of Schupp's result [5] that the two variable substitution problem is solvable in free groups. Other positive results on determining whether or not classes of equations have solutions in free products include those of Wicks [6], [7] on equations of the form $[x, y] = U$ and $x^2y^2 = U$, and Comerford and Edmunds [2] on quadratic equations.

Our second result describes solutions to equation (1), excluding some degenerate cases that will be dealt with in the next section. Here the *rank* of a solution ϕ to (1) is the rank of the subgroup $\langle x\phi, y\phi \rangle$ of G.

Theorem 2. *Suppose that $G = *_{i \in I} G_i$ is a free product of torsion-free groups G_i, $i \in I$, and that in each G_i non-trivial cyclic subgroups have finite index in their centralizers. Further, suppose that $U \in G$ with $U \notin V^{-1}G_iV$ for any $V \in G$ and $i \in I$, that $W(x, y) \in F = \langle x, y; \rangle$, and that $W(x, y) = U$ has solutions, but no solutions of rank one. Then there are finitely many solutions $\phi_1, \dots, \phi_m$ to $W(x, y) = U$ such that every solution to $W(x, y) = U$ has the form $\phi = \sigma\phi_j$ for some j with $1 \leqslant j \leqslant m$ and some $\sigma \in \mathrm{Stab}(W(x, y)) = \{\alpha \in \mathrm{Aut}(F) : (W)\alpha = W\}$.*

Note that the hypothesis in Theorem 2 on the free factors G_i that $[C_{G_i}(x) : \langle x \rangle] < \infty$ for $1 \neq x \in G_i$ implies that elements of G_i have only finitely many roots, that is, that for $x \in G_i$ there are only finitely many $y \in G_i$ with $y^n = x$ for some $n \in \mathbb{Z}$. Again, free groups and torsion-free hyperbolic groups [3, Corollary 7.2] have this property.

2 Solutions in a free factor and rank one solutions

In this section, we discuss two special cases in which it is relatively straightforward to determine whether or not equation (1) has a solution. As in our theorems, we let G be the free product of groups G_i, $i \in I$.

Suppose first that $U = V^{-1}U_0V$ with $U_0 \in G_i$ for some $i \in I$. Then equation (1) has a solution if and only if

$$W(x, y) = U_0 \tag{2}$$

has a solution, and if (2) has a solution ϕ, then it has a solution $\phi\pi_i$ with $x\phi\pi_i, y\phi\pi_i \in G_i$, where π_i is the projection of G onto G_i. Thus we may decide if equation (1) has a solution in G if we can solve the two variable substitution problem for G_i. Henceforth, then, we consider only the case that U is not in a conjugate of a free factor of G.

We next consider how to determine whether or not equation (1) has a solution of rank less than two. Plainly, (1) has a rank zero solution exactly if $U = 1$ in G. Suppose that (1) has a rank one solution, that is, that for some $V \in G$, $V \neq 1$ and $s, t \in \mathbb{Z}$ we have $V^{\sigma_x(W)s+\sigma_y(W)t} = U$, where $\sigma_x(W)$ and $\sigma_y(W)$ are the exponent sums for x and y in W. Since $\sigma_x(W)s + \sigma_y(W)t$ is a multiple of

$$\gcd(W(x, y)) = \gcd(\sigma_x(W), \sigma_y(W))$$

(where we take $\gcd(0, 0)$ to be zero), it follows that $x^{\gcd(W(x,y))} = U$ has a solution. Conversely, if $x^{\gcd(W(x,y))} = U$ has a solution $x = V$ with $1 \neq V \in G$ and if $\gcd(W(x, y)) = \sigma_x(W)s + \sigma_y(W)t$ with $s, t \in \mathbb{Z}$, then $x\phi = V^s$, $y\phi = V^t$ is a rank one solution to (1). Since determining if an equation $x^{\gcd(W(x,y))} = U$ has a solution in a free product requires only that one can solve the word problem in the free factors, the ability to solve the word problem in the free factors lets us decide whether or not (1) has a rank one solution.

3 Rank two solutions

In this section we will prove two somewhat technical lemmas concerning rank two solutions to equation (1), and then show how Theorems 1 and 2 follow from these lemmas.

We start with some notation and terminology. If $G = *_{i \in I} G_i$, and each free factor G_i has a specified finite generating set $\mathscr{A}_i$, with the sets $\mathscr{A}_i$ disjoint from one another, then $\mathscr{A} = \bigcup_{i \in I} \mathscr{A}_i$ is a generating set for G. With this understood, we shall usually suppress the distinction between a word on a generating set and the group element it defines. We use $|\cdot|$ to denote free product length. For $U, V \in G$, $U \cdot V$ indicates that there is neither cancelation nor consolidation in the product UV. An element of G is *cyclically reduced* if its length is at most one or the first and last letters of its normal form lie in different free factors of G and is *weakly cyclically reduced* if its length is at most one or the first and last letters of its normal form are not inverses.

Lemma 1. *Suppose that* $W(x, y) \in F = \langle x, y; \rangle$, $U \in G = *_{i \in I} G_i$ *with* $U \notin V^{-1} G_i V$ *for all* $V \in G$ *and* $i \in I$, *and that* $W(x, y) = U$ *has a solution of rank two but not of smaller rank. If* ϕ *is a solution to* $W(x, y) = U$, *then there are* $\alpha \in \operatorname{Aut}(F)$ *and* $\beta \in \operatorname{Inn}(G)$ *such that* $U\beta$ *is cyclically reduced and the solution* $x\alpha^{-1}\phi\beta = X$, $y\alpha^{-1}\phi\beta = Y$ *to* $W\alpha = U\beta$ *satisfies one of the following conditions*:

(a) $X \in G_i$ *for some* $i \in I$ *and either*
 (i) $Y \in G_j$ *for some* $j \in I$ *with* $j \neq i$, *or*
 (ii) $Y = Z^{-1} \cdot Y_0 \cdot Z$ *with* $Z \neq 1$, Y_0 *weakly cyclically reduced*, $XY = X \cdot Y$, *and* $YX = Y \cdot X$, *or*
 (iii) $|Y| \geqslant 2$, Y *is weakly cyclically reduced*, $YX = Y \cdot X$, *and* $|X^l Y| \geqslant |Y|$ *for all* $l \in \mathbb{Z}$, *or*

(b) $|X| \geqslant 2$, *X is weakly cyclically reduced, and either*
  (i) $Y = Z^{-1} \cdot Y_0 \cdot Z$ *with* $Z \neq 1$ *and* Y_0 *weakly cyclically reduced,* $|Y_0| \geqslant 2$, *and both* $XY = X \cdot Y$ *and* $YX = Y \cdot X$, *or*
  (ii) $|Y| \geqslant 2$, *Y is weakly cyclically reduced,* $YX = Y \cdot X$, *and there is no cancelation in any product* $Y^{\delta}X^{\varepsilon}$ *with* $\delta, \varepsilon \in \{-1, +1\}$, *or*
  (iii) $|Y| \geqslant 2$, *Y is weakly cyclically reduced,* $X = X_0 \cdot Z$, $Y = Y_0 \cdot Z$ *with* $Z \neq 1$, $|Z| \leqslant |X_0| \leqslant |Y_0|$, $YX = Y \cdot X$, *there is no cancelation in the products* $Y^{-1}X$ *or* $Y_0X_0^{-1}$ *and, if* $|Z| = |X_0| = |Y_0| = 1$, X_0, Y_0, *and Z lie in three different free factors of G.*

Proof. We will first show that there are $\alpha \in \mathrm{Aut}(F)$ and $\beta \in \mathrm{Inn}(G)$ so that $X = x\alpha^{-1}\phi\beta$ and $Y = y\alpha^{-1}\phi\beta$ satisfy one of the six conditions in the conclusion of the lemma, and then show that we may also have $U\beta$ cyclically reduced.

Among all $\alpha \in \mathrm{Aut}(F)$, $\beta \in \mathrm{Inn}(G)$, we choose α, β so that

$$|\alpha^{-1}\phi\beta| = |x\alpha^{-1}\phi\beta| + |y\alpha^{-1}\phi\beta|$$

is as small as possible and so that, subject to this, $|x\alpha^{-1}\phi\beta|$ is minimal. We note that this ensures that $X = x\alpha^{-1}\phi\beta$ is weakly cyclically reduced, for if $X = X_1^{-1} \cdot X_0 \cdot X_1$ with $X_1 \neq 1$, we could let $\beta_1 \in \mathrm{Inn}(G)$ be β followed by $g \mapsto X_1 g X_1^{-1}$ and find that $|\alpha^{-1}\phi\beta_1| \leqslant |\alpha^{-1}\phi\beta|$ and $|x\alpha^{-1}\phi\beta_1| < |x\alpha^{-1}\phi\beta|$, contrary to our choice of α and β.

Our choice of α and β dictates that if either of X and Y is in a free factor of G, then $X \in G_i$ for some $i \in I$, as in (a) of the conclusion of the lemma. Let us suppose first that this is the case. If $Y \in G_j$ for some $j \in I$, it must be that $j \neq i$ as indicated in (a)(i) since U is not in a conjugate of a free factor of G. We next show that if Y is not in a free factor of G, either (a)(ii) or (a)(iii) holds, depending on whether or not Y is weakly cyclically reduced.

If Y is not weakly cyclically reduced, let $Y = Z^{-1} \cdot Y_0 \cdot Z$ with $Z \neq 1$ and Y_0 weakly cyclically reduced. If $Z = Z_1 \cdot Z_2$ with $Z_2 \in G_i - \{1\}$, we can let β_1 be β followed by $g \mapsto Z_2 g Z_2^{-1}$, so that $x\alpha^{-1}\phi\beta_1 = Z_2 X Z_2^{-1} \in G_i$, $y\alpha^{-1}\phi\beta_1 = Z_1^{-1} Y_0 Z_1$, and $|\alpha^{-1}\phi\beta_1| < |\alpha^{-1}\phi\beta|$, contrary to our choice of α and β. Thus we have $XY = X \cdot Y$ and $YX = Y \cdot X$, as indicated in (a)(ii).

Suppose now that Y is weakly cyclically reduced. If $Y = Y_1 \cdot Y_0 \cdot Y_2$ with $Y_1, Y_2 \in G_i - \{1\}$, then conjugation by Y_1 produces a pair $X' = Y_1^{-1} X Y_1 \in G_i$, $Y' = Y_0 Y_2 Y_1$ with $|X'| + |Y'| < |X| + |Y|$, contrary to our choice of α and β. Thus either $YX = Y \cdot X$ or $Y^{-1}X = Y^{-1} \cdot X$; in the latter case, we replace α with $\alpha\gamma$ where $x\gamma = x$ and $y\gamma = y^{-1}$. Further, if $|X^l Y| < |Y|$ for some $l \in \mathbb{Z}$, we can replace α by $\alpha\eta^{-1}$ where $x\eta = x$ and $y\eta = x^l y$ to reduce $|\alpha^{-1}\phi\beta|$. We thus see that the conditions of (a)(iii) hold.

Suppose now that $|X| \geqslant 2$, as in (b) of the lemma. We cannot have $Y = Z^{-1} Y_0 Z$ with $Y_0 \in G_i$ for some $i \in I$, for then if α_1 is α followed by the interchange of x and y, and β_1 is β followed by conjugation by Z^{-1}, we have $x\alpha_1^{-1}\phi\beta_1 = Y_0$ and $y\alpha_1^{-1}\phi\beta_1 = ZXZ^{-1}$, yielding $|\alpha_1^{-1}\phi\beta_1| \leqslant |\alpha^{-1}\phi\beta|$ and $|x\alpha_1^{-1}\phi\beta_1| < |x\alpha^{-1}\phi\beta|$. We shall show that one of the sets of conditions (b)(i), (b)(ii), or (b)(iii) holds.

If, as in conclusion (b)(i) of the lemma, $Y = Z^{-1} \cdot Y_0 \cdot Z$ with $Z \neq 1$ and Y_0 weakly cyclically reduced, we have seen that $|Y_0| \geqslant 2$. If $Z = Z_1 \cdot Z_2$ with $1 \neq Z_2 \in G_i$ for some $i \in I$, X neither begins nor ends with a syllable from G_i, for then if β_1 is β followed by conjugation by Z_2^{-1}, we find that $|\alpha^{-1}\phi\beta_1| < |\alpha^{-1}\phi\beta|$, much as we did in (a)(ii). Thus $XY = X \cdot Y$ and $YX = Y \cdot X$ as stated in (b)(i).

Now suppose that Y is weakly cyclically reduced and there is no cancelation in any of the products $Y^\delta X^\varepsilon$ with $\delta, \varepsilon \in \{-1, +1\}$, as in (b)(ii) of the lemma. We cannot have consolidation in each of these four products, for if $X = X_1 \cdot X_0 \cdot X_2$ and $Y = Y_1 \cdot Y_0 \cdot Y_2$ with $X_1, X_2, Y_1, Y_2 \in G_i - \{1\}$ for some $i \in I$, we can let β_1 be β followed by conjugation by X_1 and get $|\alpha^{-1}\phi\beta_1| < |\alpha^{-1}\phi\beta|$. Thus $Y^\delta X^\varepsilon = Y^\delta \cdot X^\varepsilon$ for some $\delta, \varepsilon \in \{-1, +1\}$; replacing α with $\alpha\gamma$ where $x\gamma = x^\varepsilon$ and $y\gamma = y^\delta$, we get that $YX = Y \cdot X$ as indicated in (b)(ii).

Finally, suppose as in (b)(iii) that Y is weakly cyclically reduced and that there is cancelation in one of the products $Y^\delta X^\varepsilon$, say in YX^{-1}. (We may follow α with an automorphism $x \mapsto x^{-\varepsilon}$, $y \mapsto y^\delta$ to reduce to this case.) Let us choose $Z \neq 1$ of maximal length, and then V of maximal length, such that $X = V \cdot X_0 \cdot Z$ and $Y = V \cdot Y_0 \cdot Z$. It must be that $|V| + |Z| \leqslant |X_0|$ and $|V| + |Z| \leqslant |Y_0|$, for if, say, $|V| + |Z| > |X_0|$, then we may let α_1 be α followed by $x \mapsto x$, $y \mapsto yx$ and let β_1 be β followed by conjugation by V to get $x\alpha_1^{-1}\phi\beta_1 = X_0 ZV$, $y\alpha_1^{-1}\phi\beta = Y_0 X_0^{-1}$ and $|\alpha_1^{-1}\phi\beta_1| < |\alpha^{-1}\phi\beta|$. Also, if β_1 is as above, we find that $x\alpha^{-1}\phi\beta_1 = X_0 ZV$ and $y\alpha^{-1}\phi\beta_1 = Y_0 ZV$ and that $|\alpha^{-1}\phi\beta_1| \leqslant |\alpha^{-1}\phi\beta|$; replacing β with β_1, we may absorb V into Z and assume that $V = 1$, that is, that there is no cancelation in the product $Y^{-1}X$. Our choice of Z ensures that there is no cancelation in the product $Y_0 X_0^{-1}$. We find that X is cyclically reduced, for if $X_0 = X_1 \cdot X_2$ and $Z = Z_1 \cdot Z_2$ with $X_1, Z_2 \in G_i - \{1\}$ for some $i \in I$, we may let β_2 be β followed by conjugation by X_1 to get $x\alpha^{-1}\phi\beta_2 = X_2 Z X_1$, $y\alpha^{-1}\phi\beta_2 = X_1^{-1} Y_0 Z X_1$ with $|\alpha^{-1}\phi\beta_2| \leqslant |\alpha^{-1}\phi\beta|$ and $|x\alpha^{-1}\phi\beta_2| < |x\alpha^{-1}\phi\beta|$. We have, then, that $YX = Y \cdot X$. To establish the last assertion of (b)(iii), we note that if $|Z| = |X_0| = |Y_0| = 1$ and $X_0, Y_0 \in G_i$ for some $i \in I$, and if we let α_2 be α followed by $x \mapsto xy$, $y \mapsto y$, then $x\alpha_2^{-1}\phi\beta = X_0 Y_0^{-1}$, $y\alpha_2^{-1}\phi\beta = Y_0 Z$, and $|\alpha_2^{-1}\phi\beta| < |\alpha^{-1}\phi\beta|$. This concludes our proof that α and β can be chosen so that $X = x\alpha^{-1}\phi\beta$ and $Y = y\alpha^{-1}\phi\beta$ satisfy one of the six conditions in the conclusion of the lemma.

It remains to show that we may choose $\alpha \in \mathrm{Aut}(F)$ and $\beta \in \mathrm{Inn}(G)$ so that, in addition to having $X = x\alpha^{-1}\phi\beta$ and $Y = y\alpha^{-1}\phi\beta$ satisfy one of the six sets of conditions stated in the lemma, we have $U\beta$ cyclically reduced. We do this by replacing α with $\alpha_1 = \alpha\gamma$ where γ is a suitable element of $\mathrm{Inn}(F)$. To avoid disturbing the properties of $X = x\alpha^{-1}\phi\beta$ and $Y = y\alpha^{-1}\phi\beta$, we make a corresponding replacement of β with $\beta_1 = \beta\eta$ with $\eta \in \mathrm{Inn}(G)$ chosen so that $\alpha_1^{-1}\phi\beta_1 = \alpha^{-1}\phi\beta$. This is accomplished as follows: if γ is conjugation by $V(x, y)$ in F, we let η be conjugation by $V(X, Y) = V(x, y)\alpha^{-1}\phi\beta$ in G.

Since $W(x, y) = U$ has no solutions of rank one, no automorphic image of W is conjugate to a power of x or of y. Thus we may use an initial conjugation in F to assume that

$$W\alpha = x^{m_1} y^{n_1} \dots x^{m_k} y^{n_k} \tag{3}$$

with $k \geqslant 1$ and $m_1, n_1, \dots, m_k, n_k$ non-zero integers.

If $X = x\alpha^{-1}\phi\beta$ and $Y = y\alpha^{-1}\phi\beta$ satisfy one of (a)(i), (a)(ii), or (b)(i), then this condition and equation (3) ensure that $(W\alpha)\alpha^{-1}\phi\beta = U\beta$ is cyclically reduced. In case (a)(iii), $(W\alpha)\alpha^{-1}\phi\beta$ is cyclically reduced if $n_k > 0$ and $(y^{n_k}x^{m_1}y^{n_1}\dots x^{m_k})\alpha^{-1}\phi\beta$ is cyclically reduced if $n_k < 0$.

Suppose now that X and Y are as described in either (b)(ii) or (b)(iii). Note that $YX = Y \cdot X$ implies that $X^{-1}Y^{-1} = X^{-1} \cdot Y^{-1}$ as well. Thus if $W\alpha$ has a subword (cyclically) yx or $x^{-1}y^{-1}$, we may replace $W\alpha$ with a conjugate that begins with x and ends with y or begins with y^{-1} and ends with x^{-1} whose image under $\alpha^{-1}\phi\beta$ will be cyclically reduced. Suppose, then, that $W\alpha$ has no subwords (cyclically) yx or $x^{-1}y^{-1}$. We find that if $m_1 > 0$, then $n_k < 0$ and $m_k > 0$ and $n_{k-1} < 0$ and so forth, so that $m_1, \dots, m_k$ are all positive and $n_1, \dots, n_k$ are all negative. Likewise, if $m_1 < 0$, then $n_1 > 0$ and $m_2 < 0$ and $n_2 > 0$ and so forth, so that $n_1, \dots, n_k$ are negative and $m_1, \dots, m_k$ are positive. If Y is not cyclically reduced, we have $Y^{-1}X = Y^{-1} \cdot X$ and $X^{-1}Y = X^{-1} \cdot Y$, so that $(W\alpha)\alpha^{-1}\phi\beta$ is cyclically reduced if $m_1 > 0$ and $(y^{n_k}x^{m_1}y^{n_1}\dots x^{n_k})\alpha^{-1}\phi\beta$ is cyclically reduced if $m_1 < 0$. Likewise, if X is not cyclically reduced (which can happen only in case (b)(ii)), $YX^{-1} = Y \cdot X^{-1}$ and $XY^{-1} = X \cdot Y^{-1}$, so that $(W\alpha)\alpha^{-1}\phi\beta$ is cyclically reduced if $m_1 < 0$ and $(y^{n_k}x^{m_1}y^{n_1}\dots x^{n_k})\alpha^{-1}\phi\beta$ is cyclically reduced if $m_1 > 0$. Suppose, then, that X and Y are both cyclically reduced. If $|m_i| > 1$ for some i, we replace $W\alpha$ with a conjugate that begins and ends with x or begins and ends with x^{-1}, and find that its image under ϕ is cyclically reduced. If $|n_i| > 1$ for some i, we use a conjugate of $W\alpha$ that begins and ends with y or with y^{-1}. Otherwise, we have $|m_i| = |n_i| = 1$ for all i, so that $W\alpha = (xy^{-1})^k$ or $W\alpha = (x^{-1}y)^k$ and the original equation $W(x, y) = U$ has a rank one solution, contrary to hypothesis. This concludes the proof of Lemma 1. □

We recall that two pairs (S_1, T_1) and (S_2, T_2) of elements of a group G are *Nielsen equivalent* if there is an automorphism η of $F = \langle x, y; \rangle$ such that $\psi_2 = \eta\psi_1$ where $\psi_j : F \to G$ are defined by $x\psi_j = S_j$, $y\psi_j = T_j$ for $j = 1, 2$.

Lemma 2. *Suppose that $W(x, y) \in F = \langle x, y; \rangle$, $G = *_{i\in I}G_i$ with each G_i torsion-free, $U \in G = *_{i\in I}G_i$ with $U \notin V^{-1}G_iV$ for all $V \in G$ and $i \in I$, and that $W(x, y) = U$ has a solution of rank two but not of smaller rank.*

(1) *If non-trivial cyclic subgroups of each G_i have finite index in their centralizers, then there are only finitely many Nielsen equivalence classes of pairs (X, Y) of elements of G that satisfy one of the six sets of conditions in Lemma* 1 *and such that $U' = W'(X, Y)$ for some cyclically reduced conjugate U' of U and some $W'(x, y) \in F$ with $W' = W\alpha$ for some $\alpha \in \mathrm{Aut}(F)$.*

(2) *If each free factor G_i of G has solvable word problem, solvable conjugacy problem, and effective extraction of roots, then we can effectively determine whether or not there is a word $W'(x, y) \in F$ with $W'(x, y) = W(x, y)\alpha$ for some $\alpha \in \mathrm{Aut}(F)$ and with $W'(X, Y) = U'$ for some cyclically reduced conjugate U' of U and some pair (X, Y) satisfying one of the six sets of conditions in Lemma* 1.

Proof. We shall seek to identify, for each cyclically reduced conjugate U' of U and for each of the six sets of conditions of Lemma 1, pairs (X, Y) satisfying the given conditions and with $U' = W'(X, Y)$ for some automorphic image $W'(x, y)$ of $W(x, y)$ in F. In each case, we begin with the possibilities for $W'(x, y)$ and then identify the pairs (X, Y) satisfying our conditions that would give $W'(X, Y) = U'$. We shall show that under the hypotheses of both (1) and (2), there are only finitely many possibilities for $W'(x, y)$, up to action by $\mathrm{Aut}(F)$, and that under the hypotheses of (1), there are only finitely many Nielsen equivalence classes of solution pairs (X, Y). Initially, we use only that $W'(x, y)$ is not a power of a primitive element of F and that $\gcd(W(x, y)) \neq 1$; once we have a list of candidates for $W'(x, y)$, we may use Whitehead's Theorem [1, Theorem IV.1.10] to eliminate those that are not automorphic images of $W(x, y)$.

For each of the following cases, we let

$$W'(x, y) = x^{m_1} y^{n_1} \dots x^{m_k} y^{n_k}$$

with $k \geqslant 1$, all m_i and n_i non-zero except possibly m_1 and n_k and, if $m_1 = 0$ or $n_k = 0$, $k \geqslant 2$. We then find pairs (X, Y) satisfying the appropriate conditions and values of $k, m_1, \dots, m_k$, and $n_1, \dots, n_k$ such that $W'(X, Y) = U'$.

We note that under our hypotheses, $\langle X = x\phi, Y = y\phi \rangle$ is a rank two subgroup of the torsion-free free product G and $\langle X, Y \rangle$ is not contained in a conjugate of a free factor of G. It follows from the Kurosh Subgroup Theorem [1, Theorem IV.1.10] and the Grushko–Neumann Theorem [1, Corollary IV.1.9] that $\langle X, Y \rangle$ is free, and so U' and the pair (X, Y) uniquely determine $W'(x, y)$.

Case (a)(i). If (X, Y) satisfies (a)(i) of Lemma 1, $|U'|$ must be even and the syllables of its normal form must alternate between two free factors of G. If this is so and $U' = S_1 \cdot T_1 \cdot \dots \cdot S_k \cdot T_k$ with $S_1, \dots, S_k \in G_i - \{1\}$ and $T_1, \dots, T_k \in G_j - \{1\}$ for some $i, j \in I$ with $i \neq j$, our hypotheses in both (1) and (2) tell us that there are only finitely many values $x = X \in G_i$, $y = Y \in G_j$, and non-zero integers $m_1, n_1, \dots, m_k, n_k$ such that $x^{m_1} = S_1, \dots, x^{m_k} = S_k$ in G_i and $y^{n_1} = T_1, \dots, y^{n_k} = T_k$ in G_j. (It is possible, of course, that there are no solutions.) The hypotheses of (2) of the lemma allow us to effectively calculate the solution pairs (X, Y) and associated values of $m_1, n_1, \dots, m_k, n_k$, and hence the possibilities for $W'(x, y)$ in (2).

Case (a)(ii). Next suppose that (X, Y) is as in (a)(ii) of Lemma 1. It follows that

$$U' = X^{m_1} \cdot Z^{-1} \cdot Y_0^{n_1} \cdot Z \cdot \dots \cdot X^{m_k} \cdot Z^{-1} \cdot Y_0^{n_k} \cdot Z. \tag{4}$$

If Y_0 is in a free factor G_j of G, then

$$|U'| = 2k(|Z| + 1) \tag{5}$$

if $m_1 \neq 0 \neq n_k$; if one of m_1 and n_k is zero, cyclic reduction of U' implies that $m_1 = n_k = 0$ and so $|U'| = 2(k-1)(|Z| + 1)$, with $k \geqslant 2$. (In the rest of the proof, we shall generally present only the case that $m_1 \neq 0 \neq n_k$, recognizing that slight adjustments are required in some equations and inequalities if this is not the case.) For each of the finitely many solutions of equation (5), we use the values of k and $|Z|$ in equation (4) to identify Z, and we are left to solve systems of equations $x^{m_1} = S_1, \dots, x^{m_k} = S_k$ in G_i and $y^{n_1} = T_1, \dots, y^{n_k} = T_k$ in G_j as in Case (a)(i). As before, the hypotheses of (1) tell us that there are only finitely many solution pairs (X, Y) and the hypotheses of (2) allow us to effectively calculate these pairs and their associated words $W'(x, y)$.

If Y_0 is not in a free factor of G, then $|Y_0| \geqslant 2$ and

$$|U'| = k(1 + 2|Z|) + (|n_1| + \cdots + |n_k|)|Y_0| \tag{6}$$

if Y_0 is cyclically reduced and

$$|U'| = k(2 + 2|Z|) + (|n_1| + \cdots + |n_k|)(|Y_0| - 1) \tag{7}$$

if Y_0 is not cyclically reduced. For each of the finitely many sets of values for k, $|Y_0|$, $|Z|$, and $n_1, \dots, n_k$ satisfying equation (6) or (7), we see if there are Z and Y_0 (cyclically reduced or not as appropriate) for which equation (4) might hold. When there are such Z and Y_0, the determination of $x = X \in G_i$ and $W'(x, y)$ reduces to solving a system $x^{m_1} = S_1, \dots, x^{m_k} = S_k$ in G_i as before.

Case (a)(iii). Now suppose that (X, Y) is as described in (a)(iii) of Lemma 1. If $XY = X \cdot Y$, we set $Z = 1$ and $Y_0 = Y$ and find that equation (6) or (7) holds, depending on whether or not Y is cyclically reduced; our argument then proceeds just as in the previous case. Suppose, then, that there is consolidation in the product XY. By the conditions of (a)(iii), we have $Y = Y_1 \cdot Y_2$ with $Y_1 \in G_i$ but $Y_1 \notin \langle X \rangle$, the cyclic subgroup generated by X. Thus

$$|U'| = (|n_1| + \cdots + |n_k|)|Y|. \tag{8}$$

We now consider each of the finitely many solutions for $|Y|$, k, and $n_1, \dots, n_k$ in equation (8). If some $|n_j| \geqslant 2$ for some j, the forms of $W'(x, y)$ and U' uniquely determine Y, and we are left to solve a system of equations $x^{m_1} = S_1, \dots, x^{m_k} = S_k$ in G_i as before.

We suppose, then, that $|n_1| = \cdots = |n_k| = 1$. Here we must have $k \geqslant 2$, for otherwise equation (1) would have a rank one solution. Writing $Y = Y_1 \cdot Y_2$ with $Y_1 \in G_i - \{1\}$ as before, we see that the forms of $W'(x, y)$ and U' determine Y_2, but not Y_1, for Y_1 consolidates with the powers of X in U'. Our task is to find the sets of values of the exponents $m_1, \dots, m_k$ on x in $W'(x, y)$ for which there will be Y_1 and X in G_i that will complete a solution (X, Y) to $W'(x, y) = U'$ as described in (a)(iii) of Lemma 1.

Let us consider the exponents m_j to be independent integer-valued variables, and say that m_j is of *type* (δ, ε) if $y^{\delta} x^{m_j} y^{\varepsilon}$ is a subword (cyclically) of $W'(x, y)$. Note that $W'(x, y)$ has exponents of type $(1, -1)$ if and only if it has exponents of type $(-1, 1)$.

Suppose that $W'(x, y)$ has exponents of types $(1, -1)$ and $(-1, 1)$. Matching the image of $W'(x, y)$ with the syllables of U', we get an equation

$$x^{m_j} = S_j \tag{9}$$

for each exponent m_j of type $(1, -1)$, and an equation

$$y_1^{-1} x^{m_j} y_1 = (y_1^{-1} x y_1)^{m_j} = S_j \tag{10}$$

for each exponent m_j of type $(-1, 1)$, where $S_j \in G_i$ in each case. Suppose that there is a common solution $x = X \in G_i$, $m_j = a_j \in \mathbb{Z}$ to the equations (9), and also a common solution $y_1^{-1} x y_1 = X' \in G_i$, $m_j = a_j \in \mathbb{Z}$ to the equations (10). Under the hypotheses of (2), we may see if any of the finitely many solutions X' to the equations (10) is conjugate to any of the finitely many solutions X to the equations (9) to decide whether there is a word $W'(x, y)$ as described in (2) of the lemma.

We now prove (1) of the lemma in this case. Let us suppose that a solution X to the equations (9) is conjugate to a solution X' to the equations (10), with $X' = T^{-1} X T$ and $T \in G_i$. Then every solution to $y_1^{-1} X y_1 = X'$ has the form $y_1 = CT$ with $C \in C_{G_i}(X)$. If $C_1, \ldots, C_k$ is a system of coset representatives for $\langle X \rangle$ in $C_{G_i}(X)$, then we have $y_1 = X^s C_t T$ with $1 \leqslant t \leqslant k$ and $s \in \mathbb{Z}$. For each t with $1 \leqslant t \leqslant k$, this produces an infinite set of solution pairs satisfying our hypotheses, $(X, X^s C_t T Y_2)$ with $s \in \mathbb{Z}$, but these are Nielsen equivalent to one another. Hence the conclusion of (1) of the lemma holds in this case.

Suppose now that $W'(x, y)$ has exponents of type $(1, 1)$ but not of type $(-1, -1)$. There must be more than one exponent of type $(1, 1)$, for otherwise the exponent sum on y in W' would be one and $W'(x, y) = U'$ would have a rank one solution. For each exponent m_j of type $(1, 1)$, we get an equation

$$x^{m_j} y_1 = S_j \tag{11}$$

with $S_j \in G_i$. Rewriting equation (11) as $y_1 = x^{-m_j} S_j$, we see that we must have

$$x^{m_j - m_l} = S_j S_l^{-1} \tag{12}$$

for all exponents m_j, m_l of type $(1, 1)$. If there is a common solution $x = X$ for the equations (12), as well as any equations (9) and (10), the exponents of type $(1, 1)$ will have values $m_j = a_j + m$ with $a_j \in \mathbb{Z}$ and $m \in \mathbb{Z}$ arbitrary. While this yields infinitely many possibilities for (X, Y) and the associated words $W'(x, y)$, we see that these pairs (X, Y) are all Nielsen equivalent under maps $x \mapsto x$, $y \mapsto x^m y$, and that the associated words $W'(x, y)$ are all in the same orbit under $\mathrm{Aut}(F)$. Thus the conclusions of both (1) and (2) of the lemma follow.

Now suppose that $W'(x, y)$ has exponents of type $(-1, -1)$ but not of type $(1, 1)$. As before, there must be more than one exponent m_j of type $(-1, -1)$, and for each such exponent we have an equation

$$y_1^{-1} x^{m_j} = S_j \tag{13}$$

with $S_j \in G_i$. Rewriting equation (13) as $y_1 = x^{m_j} S_j^{-1}$, we find that we must have

$$x^{m_j - m_l} = S_l^{-1} S_j \tag{14}$$

for all exponents m_j, m_l of type $(-1,-1)$. The determination of possibilities for solution pairs (X, Y) and associated words $W'(x, y)$ is then just as in the previous case.

Finally, suppose that $W'(x, y)$ has exponents of both type $(1,1)$ and type $(-1,-1)$. It follows that $W'(x, y)$ must have exponents of types $(1,-1)$ and $(-1,1)$ as well. Suppose that $x = X \in G_i$, $y_1 = Y_1 \in G_i$ is a common solution to all equations (9) and (10), as well as any equations (12) and (14). Using the previously noted fact that the general solution for y_1 in equations (11) and (13) is $y_1 = CY_1$ with $C \in C_{G_i}(X)$, we see that equations (11) and (13) require that $C = X^{-m_j} S_j Y_1^{-1}$ for all exponents m_j of type $(1,1)$ and that $C = X^{m_l} S_l^{-1} Y_1^{-1}$ for all exponents m_l of type $(-1,-1)$. We thus get the conditions

$$X^{m_j+m_l} = S_l S_j \tag{15}$$

for all pairs of exponents m_j of type $(1,1)$ and m_l of type $(-1,-1)$. As before, we see that while the equations (15) may yield infinitely many possible solution pairs (X, Y) and associated words $W'(x, y)$, these pairs (X, Y) are all Nielsen equivalent and the words $W'(x, y)$ are all in the same orbit under $\mathrm{Aut}(F)$, and so the conclusions of (1) and (2) of the lemma follow.

Case (b)(i). Suppose now that (X, Y) is as in part (b)(i) of Lemma 1. Here we find that

$$|U'| = |X^{m_1}| + \cdots + |X^{m_k}| + |Y_0^{n_1}| + \cdots + |Y_0^{n_k}| + 2k|Z|$$

with $|X^{m_i}| = |m_i|\,|X|$ if X is cyclically reduced and $|X^{m_i}| = |m_i|(|X| - 1) + 1$ if X is not cyclically reduced, and with similar equations for $|Y_0^{n_i}|$. In any event, we have

$$|U'| \geqslant (|m_1| + \cdots + |m_k|)(|X| - 1) + (|n_1| + \cdots + |n_k|)(|Y_0| - 1) + 2k(|Z| + 1). \tag{16}$$

For each of the finitely many sets of values for k, $|X|$, $|Y_0|$, $|Z|$, and $m_1, n_1, \ldots, m_k, n_k$ which satisfy equation (16), the forms of $W'(x, y)$ and U' determine the possible values of X, Y_0, and Z uniquely.

Case (b)(ii). Here we take (X, Y) to be as described in (b)(ii) of Lemma 1. Allowing for possible consolidations between powers of X and Y, we find that

$$|U'| \geqslant |X^{m_1}| + |Y^{n_1}| + \cdots + |X^{m_k}| + |Y^{n_k}| - (2k - 1). \tag{17}$$

Further, we have $|X^{m_j}| \geqslant |m_j|(|X| - 1) + 1$ and $|Y^{n_j}| \geqslant |n_j|(|Y| - 1) + 1$, except that possibly $|X^{m_1}| = 0$ or $|Y^{n_k}| = 0$. Since we may replace $2k - 1$ on the right side of inequality (17) by $2k - 2$ if one of $|X^{m_1}|$, $|Y^{n_k}|$ is zero and by $2k - 3$ if $|X^{m_1}| = |Y^{n_k}| = 0$, we find that in all cases

$$|U'| \geqslant (|m_1| + \cdots + |m_k|)(|X| - 1) + (|n_1| + \cdots + |n_k|)(|Y| - 1) + 1. \tag{18}$$

We consider each of the finitely many solutions k, $|X|$, $|Y|$, and $m_1, n_1, \ldots, m_k, n_k$ to inequality (18).

Let s and t be the first and last letters of $W'(x, y)$ (which are determined by our solution to (18)) and let S and T be their images under the substitution $x = X$, $y = Y$. From the condition in (b)(ii) that there is no cancelation in the products XX, YY, or any of $Y^{\delta}X^{\varepsilon}$ with $\delta, \varepsilon \in \{-1, +1\}$, and knowing both $|X|$ and $|Y|$, comparison with U' reveals all of S and T except possibly the last syllable of S and the first syllable of T. We know that $t \neq s^{-1}$ because U' is cyclically reduced. If $s = t$, then all of $S = T$ is known and, knowing one of X and Y, the equation $W'(X, Y) = U'$ will tell us the other.

We are left with the case that $t \neq s^{\pm 1}$, so that one of s, t is $x^{\pm 1}$ and the other is $y^{\pm 1}$. If $W'(x, y)$ has any subwords ss, $s^{-1}s^{-1}$, st^{-1}, or ts^{-1}, comparison of $W'(X, Y)$ with U' will reveal the last syllable of S, and then the first syllable of T, and so tell us X and Y. Similarly, if $W'(x, y)$ has subwords tt, $t^{-1}t^{-1}$, $s^{-1}t$, or $t^{-1}s$, we will learn the last syllable of T and therefore X and Y. This leaves us only with $W' = (st)^k$, but in that event equation (1) has a rank one solution.

Case (b)(iii). Finally, suppose that (X, Y) satisfies the conditions of (b)(iii) of Lemma 1. These conditions tell us that we can have cancelation (of a Z with a Z^{-1}) in a product $X^{m_i}Y^{n_i}$ only if $m_i > 0$ and $n_i < 0$, and cancelation (of a Z^{-1} with a Z) in a product $Y^{n_i}X^{m_{i+1}}$ only if $n_i > 0$ and $m_{i+1} < 0$. Allowing for possible cancelations and consolidations, then, we have

$$|U'| \geqslant |X^{m_1}| + |Y^{n_1}| + \cdots + |X^{m_k}| + |Y^{n_k}| - 2k|Z| - (2k - 1). \tag{19}$$

The conditions of (b)(iii) also tell us that $|X^{m_j}| = |m_j|(|X_0| + |Z|)$ and that $|Y^{n_j}| \geqslant |n_j|(|Y_0| + |Z| - 1) + 1$. Combining these and inequality (19), we get

$$\begin{aligned} |U'| \geqslant{} & (|m_1| + \cdots |m_k|)|X_0| + (|n_1| + \cdots + |n_k|)(|Y_0| - 1) \\ & + (|m_1| + \cdots + |m_k| + |n_1| + \cdots |n_k| - 2k)|Z| - k + 1. \end{aligned} \tag{20}$$

We shall now show that k is bounded in terms of $|U'|$. By hypothesis,

$$|m_1| + \cdots + |m_k| \geqslant k \quad \text{and} \quad |n_1| + \cdots + |n_k| \geqslant k,$$

so that if $|Y_0| \geqslant 2$, then inequality (20) implies that $|U'| \geqslant k + 1$. Suppose, therefore, that $|Y_0| = 1$. We then have $|Z| = |X_0| = 1$ as well. Further, we find that Y is cyclically reduced and that $X_0 Y_0^{-1} = X_0 \cdot Y_0^{-1}$, so that

$$\begin{aligned} |U'| &\geqslant |X^{m_1}| \ldots |X^{m_k}| + |Y^{n_1}| + \cdots + |Y^{n_k}| - 2k|Z| \\ &= 2(|m_1| + \cdots + |m_k| + |n_1| + \cdots + |n_k|) - 2k \\ &\geqslant 2k. \end{aligned}$$

Given the bound on k, inequality (20) establishes bounds on $|X_0|$, $|Y_0|$, $|m_1|, \ldots, |m_k|$, and $|n_1|, \ldots, |n_k|$. Inequality (20) also bounds $|Z|$ unless the number of cancelations in products $X^{m_i}Y^{n_i}$ and $Y^{n_i}X^{n_{i+1}}$ is k and

$$|m_1| = \cdots = |m_k| = |n_1| = \cdots = |n_k| = 1;$$

but in this event $W' = (xy^{-1})^k$, and $W'(x, y) = U'$ has a rank one solution.

Now each of the finitely many sets of possible values for $k, m_1, \ldots, m_k, n_1, \ldots, n_k$, $|X_0|$, $|Y_0|$, and $|Z|$ determines the word $W'(x, y)$, and our task is to see if there is a solution to $W'(x, y) = U'$ satisfying the conditions of (b)(iii) of Lemma 1 and with the given values for $|X_0|$, $|Y_0|$, and $|Z|$. To do this, we further distinguish cases according to whether or not there is consolidation in multiplying non-inverse elements of the set $\mathscr{S} = \{X_0, X_0^{-1}, Y_0, Y_0^{-1}, Z, Z^{-1}\}$. (Our hypotheses rule out consolidation in several instances.) We shall show that in each case, our data on $W'(x, y)$, $|X_0|$, $|Y_0|$, and $|Z|$ and when products of elements of $\mathscr{S}$ give consolidation will allow us to uniquely identify (X, Y) if such a pair exists.

Information on $W'(x, y)$, $|X_0|$, $|Y_0|$, $|Z|$, and when consolidations occur among products of elements of $\mathscr{S}$ allows us to identify the syllables of U' as coming from $X_0^{\pm 1}$, $Y_0^{\pm 1}$, $Z^{\pm 1}$, or the consolidation of first and last syllables in products of these. As a special case, we note that if $|Y_0| = 1$, there is no consolidation in products of non-inverse elements of $\mathscr{S}$, and so X_0, Y_0, and Z are apparent. We shall show that if $|Y_0| \geqslant 2$, examination of the images of the first and last letters of $U' = W'(X, Y)$ will tell us two of X_0, Y_0, and Z; from this, it is easy to determine the third.

Let s and t be the first and last letters of $W'(x, y)$. Since $U' = W'(X, Y)$ is cyclically reduced, we cannot have $t = s^{-1}$, $s = y^{-1}$ and $t = x$, or $s = x^{-1}$ and $t = y$. By a case analysis, we may see that if there is no cancelation in the images of the first and second letters of $W'(x, y)$ or in the images of the next-to-last and last letter of $W'(x, y)$, we can determine at least two of X_0, Y_0, and Z. For example, suppose that $W'(x, y)$ begins with x, but not xy^{-1}, and ends with y. From examining initial and terminal subwords of U', then, we learn all of X_0Z except possibly its last syllable and all of Y_0Z except possibly its first syllable. Since $|Y_0| \geqslant 2$, then, we know Z, and we can then find X_0 from our knowledge of X_0Z. Other cases of this type are similar.

Finally, suppose that there is cancelation between the images of the first two or the last two letters of $W'(x, y)$. We examine the case that $W'(x, y)$ begins with xy^{-1}; other cases are similar. Looking at an initial segment of U' we find the value of $X_0Y_0^{-1}$, except possibly its last syllable. Now t is one of x, y, and y^{-1}. If $t = x$, we look at a terminal segment of U' to learn all of X_0Z except possibly its first syllable; but we know the first syllable of X_0 from our information on $X_0Y_0^{-1}$, so that X_0 and Z are determined. If $t = y$, we know Y_0Z except possibly its first syllable, and so we know Y_0 and Z. The case that $t = y^{-1}$ is similar, unless $W'(x, y)$ ends in xy^{-1}. In this event, the initial and terminal subwords of U' that are images of the copies of xy^{-1} together tell us $X_0Y_0^{-1}$, but not necessarily the terminal letters of X_0 or Y_0. We may learn the terminal letters of X_0 and Y_0 from an image of a two-letter subword of $W'(x, y)$ unless every copy of $x^{\pm 1}$ and $y^{\pm 1}$ is part of a subword $(xy^{-1})^{\pm 1}$ of $W'(x, y)$. But if this is the case, $W'(x, y) = (xy^{-1})^k$ and $W'(x, y) = U'$ has a rank one solution. □

Proof of Theorem 1. This follows directly from Lemma 1 and (2) of Lemma 2. □

Proof of Theorem 2. Assume the hypotheses of Theorem 2, which are also those of (1) of Lemma 2. Let $(X_1, Y_1), \dots, (X_k, Y_k)$ be representatives of the Nielsen equivalence classes described in (1) of Lemma 2, and for each j, $1 \leqslant j \leqslant k$, let $\alpha_j \in \mathrm{Aut}(F)$ and $W_j'(x, y) \in F$ with $W_j'(x, y) = W(x, y)\alpha_j$ and with $W_j(X_j, Y_j) = U_j'$, a cyclically reduced conjugate of U. Define $\psi_j : F \to G$ by $x\psi_j = X_j$, $y\psi_j = Y_j$ for $1 \leqslant j \leqslant k$.

Now suppose that $U = U_0^N$ with N a positive integer and U_0 not a proper power in G. Since U_j' is a cyclically reduced conjugate of U for $1 \leqslant j \leqslant k$, it follows that U_0 factors as $U_0 = S_j \cdot T_j$ with $U_j' = (T_j S_j)^N$. For $0 \leqslant i \leqslant N-1$, we define $\beta_{j,i} \in \mathrm{Inn}(G)$ by $(g)\beta_{i,j} = (T_j S_j)^{-i} S_j^{-1} g S_j (T_j S_j)^i$.

We now let $\phi_{j,i} = \alpha_j \psi_j \beta_{j,i}^{-1}$ for $1 \leqslant j \leqslant k$ and $0 \leqslant i \leqslant N-1$. It is apparent that $\{\phi_{j,i} : 1 \leqslant j \leqslant k, 0 \leqslant i \leqslant N-1\}$ is a set of solutions to $W(x, y) = U$. It remains to show that for any solution ϕ to $W(x, y) = U$ we have $\phi = \sigma\phi_{j,i}$ for some $\sigma \in \mathrm{Stab}(W(x, y))$ and some j with $1 \leqslant j \leqslant k$ and i with $0 \leqslant i \leqslant N-1$.

Suppose, then, that ϕ is a solution to $W(x, y) = U$. By Lemma 1 and (1) of Lemma 2, there are $\alpha \in \mathrm{Aut}(F)$ and $\beta \in \mathrm{Inn}(G)$ such that the pair (X, Y) with $X = x\alpha^{-1}\phi\beta$ and $Y = y\alpha^{-1}\phi\beta$ is Nielsen equivalent to (X_j, Y_j) for some j with $1 \leqslant j \leqslant k$. Suppose that $\eta \in \mathrm{Aut}(F)$ with $\alpha^{-1}\phi\beta = \eta\psi_j$, with ψ_j as defined above. There is an i with $0 \leqslant i \leqslant N-1$ such that $\beta = \gamma\beta_{j,i}$ and $\gamma \in \mathrm{Inn}(G)$ is conjugation by U^m for some integer m. Thus if $\hat{\gamma} \in \mathrm{Inn}(F)$ is conjugation by $(W(x, y))^n$, we see that

$$\phi = \alpha\eta\psi_j\beta^{-1} = \alpha\eta\psi_j\beta_{j,i}^{-1}\gamma^{-1} = (\alpha\eta\alpha_j^{-1})\phi_{j,i}\gamma^{-1} = (\hat{\gamma}^{-1}\alpha\eta\alpha_j^{-1})\phi_{j,i}$$

with $\hat{\gamma}^{-1}\alpha\eta\alpha_j^{-1} \in \mathrm{Stab}(W(x, y))$. This concludes the proof of Theorem 2. □

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